D'Alembert's Ratio Test

Suppose $\sum u_n$ is a series of positive real numbers and let $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l$.

Then: (1) $\sum u_n$ convergent if $l < 1$

(2) $\sum u_n$ divergent if $l > 1$

$= 1$ (Test fails)

Ex: Show that $1 + \frac{3}{2!} + \frac{5}{3!} + \frac{7}{4!} + \cdots$ is conv.

Ans. here $u_n = \frac{2n-1}{n!}$ (individual term)

$$\lim \frac{u_{n+1}}{u_n} = \lim \frac{2n+1}{(n+1)!} \cdot \frac{n!}{(2n-1)} = \lim \frac{2n+1}{(n+1)(2n-1)} = 0$$

$\Rightarrow \sum u_n$ conv. (Proved)

H.W. Find whether following series conv/div:

(1) $\sum \frac{x^n}{n}$ $\longrightarrow \frac{x^{n+1}}{x^n} \cdot \frac{n+1}{n} \longrightarrow x < 1$

(2) $\sum \frac{n^2-1}{n^2+1} x^n$

(3) $\sum \frac{1}{n}$ $\lim \frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n}{n+1} =$

(4) $\sum \frac{1}{n^2}$

(5) $\sum \frac{n^n}{n!}$

(6) $x + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\frac{3}{4}\frac{x^5}{5} + \cdots$

Cauchy's Root Test

$\sum u_n \simeq \sum f^n$

Suppose $\sum u_n$ is series of +ve $\mathbb{R}$.

infinite GP

Let $\lim_{n \to \infty} u_n^{1/n} = l$. Then

$\lim u_n \simeq l^n$

(1) $\sum u_n$ conv if $l < 1$

$\sum u_n \simeq \sum 0^n$

(2) $\sum u_n$ divg if $l > 1$

Ex: Show that $\frac{1}{2} + \frac{1}{3^2} + \frac{1}{4^3} + \cdots$ is conv

Ans. $u_n = \frac{1}{(n+1)^n}$. $\lim_{n \to \infty} u_n^{1/n} = \lim_{n \to \infty} \frac{1}{n+1} = 0 < 1$

$\Rightarrow \sum u_n$ conv (proved)

H.W. Conv / divg :

Ans

(1) $\sum \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$ $\quad \frac{1}{\left(1+\frac{1}{n}\right)^{n}} \to \frac{1}{e} < 1$ $\quad$ C

(2) $\sum \frac{n^{n^2}}{(1+n)^{n^2}}$ $\quad \to \frac{n^n}{(1+n)^n} = \frac{1}{\left(1+\frac{1}{n}\right)^n}$ $\quad$ C

(3) $\frac{1^2 \cdot 2^2}{1!} + \frac{2^2\, 3^2}{2!} + \frac{3^2 \cdot 4^2}{3!} + \cdots$ $\quad \lim \frac{(n+1)^2}{n-1}\,\frac{1}{n}$ $\quad$ C

$\frac{n^2(n+1)^2}{n!} \quad \frac{(n-1)!}{(n-1)^2 n^2}$ $\quad$ C

(4) $1 + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \cdots$ $\quad$ C

(5) $\frac{1^2}{2^2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} + \cdots$ $\quad$ D

Imp

(6) $\frac{1}{\log 2} + \frac{1}{\log 3} + \frac{1}{\log 4} + \cdots\cdots$ D

(7) $1 + \frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{2^2 \cdot 3^2} + \frac{1}{2^2 \cdot 3^3} + \cdots\cdots$ C

(8) $\left(\frac{2^2}{1^2} - \frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3} - \frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4} - \frac{4}{3}\right)^{-3} + \cdots\cdots$ C

Hint: Use $\log(1+x) < x$

H.W.:

(1) For $c \in \mathbb{R}$, let $u_n = \dfrac{\left(1+\frac{c}{n}\right)^{n^2}}{\left(3-\frac{1}{n}\right)^{n}}$.

Then find values of c for which $\sum u_n$ converges:

(a) $\log 6 < c < \log 9$

(b) $c < \log 3$

(c) $\log 9 < c < \log 12$

(d) $\log 3 < c < \log 6$

General form of Ratio & Root Test

Ratio Test: Suppose $\sum u_n$ is a series of +ve R.

Let $\overline{\lim_{n\to\infty}} \dfrac{u_{n+1}}{u_n} = R$, $\underline{\lim_{n\to\infty}} \dfrac{u_{n+1}}{u_n} = r$

Then $\sum u_n$ conv. if $R < 1$ and divg if $r > 1$

Root Test: Suppose $\sum u_n$ is series of +ve $\mathbb{R}$.

Let $\overline{\lim_{n\to\infty}} u_n^{1/n} = r$.

Then $\sum u_n$ conv. if $r < 1$ and divg if $r > 1$

Example : Conv/div :

$$a+b+a^2+b^2+a^3+b^3+\cdots \quad [0<a<b<1]$$

Ans of the form $\sum u_n$, where

$u_{2n}=b^n$, $u_{2n+1}=a^{n+1}$, $u_{2n-1}=a^n$

Now, $\lim \dfrac{u_{2n}}{u_{2n-1}} = \lim\limits_{n\to\infty}\left(\dfrac{b}{a}\right)^n = \infty$

$\lim \dfrac{u_{2n+1}}{u_{2n}} = \lim\limits_{n\to\infty} a\left(\dfrac{a}{b}\right)^n = 0$

BUT

$\overline{\lim}\, u_n^{1/n} = \sqrt{b}$

$\lim (u_{2n})^{1/n} = \lim b^{\frac{n}{2n}} = \sqrt{b}$

$\lim (u_{2n+1})^{1/2n+1} = \sqrt{a}$

$\lim (u_{2n-1})^{1/2n-1} = \sqrt{a}$

Hence, $\overline{\lim}\dfrac{u_{n+1}}{u_n} = \infty$ and $\underline{\lim}\dfrac{u_{n+1}}{u_n} = 0$

$\Rightarrow$ Ratio Test gives NO decision, But

$\overline{\lim}\, u_n^{1/n} = \sqrt{b} \Rightarrow \sum u_n$ conv (proved)

H.W.

Conv / div :

(1) $\frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots$

(2) $\frac{1}{2} + 2 + \frac{1}{2^2} + 2^2 + \frac{1}{2^3} + 2^3 + \cdots$

Raabe's Test

$\cancel{n}\left(\text{increment}\right) \simeq \frac{m}{n} \cdot u_n$

Suppose $\sum u_n$ be a series of +ve real and let $\lim_{n \to \infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = m$. Then

$\lim n^4 \frac{1}{n^3}$

$\sum u_n$ is convergent if $m > 1$

$\sum u_n$ is divergent if $m < 1$

☆ If ratio test fails, then we go for Raabe's Test

Imp

Example

Q: Consider $x + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{x^5}{5} + \cdots$

Then whether the series conv/div at $x=1$?

Ans: The given series can be written as : $\sum u_n$

where $u_n = \dfrac{1\cdot 3\cdot 5\cdot \ldots (2n-3)\, x^{2n-1}}{2\cdot 4\cdot 6 \cdots (2n-2)(2n-1)}$

Now, $\lim_{n\to\infty} \dfrac{u_{n+1}}{u_n} = \lim_{n\to\infty} \dfrac{(2n-1)^2}{2n(2n+1)} x^2 = x^2$
$= 1$ (at $x=1$)

$\Rightarrow$ Ratio test fails $\Rightarrow$ We go for Raabe's Test.

$\lim_{n\to\infty} n\left(\dfrac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} n\left[\dfrac{2n(2n+1)}{(2n-1)^2} - 1\right] \stackrel{H.W}{=} \dfrac{3}{2} > 1$

$\Rightarrow$ by Raabe's Test, $\sum u_n$ convergent.

H.W. Find whether the following series conv/div

[Find the ranges of unknown (x, a, b) for which the series conv/div.

(1) $\frac{3}{7}x + \frac{3\cdot 6}{7\cdot 10}x^2 + \frac{3\cdot 6\cdot 9}{7\cdot 10\cdot 13}x^3 + \cdots$

(2) $\frac{a}{b} + \frac{1+a}{1+b} + \frac{(1+a)(2+a)}{(1+b)(2+b)} + \cdots$

(3) $\frac{2}{3} + \frac{2\cdot 4}{3\cdot 5}x + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}x^2 + \cdots$

(4) $1 + \frac{2^2}{3^2}x + \frac{2^2\cdot 4^2}{3^2\cdot 5^2}x^2 + \frac{2^2\cdot 4^2\cdot 6^2}{3^2\cdot 5^2\cdot 7^2}x^3 + \cdots$

H.W.

Imp Prove that, the series for $a, b, c > 0$

$$\frac{a}{b} + \frac{a(a+c)}{b(b+c)} + \frac{a(a+c)(a+2c)}{b(b+c)(b+2c)} + \cdots$$

is conv if $b > a + c$

and div. if $b \leq a + c$